"HIGHER" STRUCTURES

– IN –

MODULI THEORY

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MODULI THEORY

PLAN

- Introduction
- D-critical loci
- Derived algebraic geometry
- Shifted symplectic structures
- D-manifolds (depending on time)

INTRODUCTION

X : Calabi–Yau 3-fold

Y : moduli space of (stable) sheaves Q on X of fixed determinant

$$\begin{tikzcd}
 & \mathbb{Q} \arrow[d] & \\
 & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X & & Y
\end{tikzcd}$$

universal family

$\rightsquigarrow$ perfect obstruction theory

$$E = Rq_* R\mathcal{H}om(\mathbb{Q},\mathbb{Q})_0[2] \xrightarrow{\ \phi\ } \mathbb{L}_Y^{\geq -1}$$

trace $= 0$

Serre duality + $K_X \cong \mathcal{O}_X$, $\dim X = 3$

imply $\quad E \xrightarrow[\Theta]{\sim} E^\vee[1]$, $\quad \Theta^\vee = \Theta[1]$.

so POT is symmetric (Behrend).

Fiberwise at $[Q] \in Y$

$$0 = \mathrm{Ext}^0(Q,Q)_0 \xrightarrow{\sim} \mathrm{Ext}^3(Q,Q)_0^\vee$$

$$\mathrm{Ext}^1(Q,Q)_0 \xrightarrow{\sim} \mathrm{Ext}^2(Q,Q)_0^\vee$$

Local picture

$$\begin{tikzcd} & \Omega_V \arrow[d] \\ \text{section } \omega \arrow[ur, dashed, bend left] & V \arrow[l, dashed, bend left] \end{tikzcd}$$

section ω = 1-form; Ω_V cotangent bundle; V smooth scheme

Then $Y \rightsquigarrow \mathcal{U} = (\omega = 0) \subseteq V$

$$\begin{tikzcd} E \arrow[r, squiggly] \arrow[d] & T_V|_{\mathcal{U}} \arrow[r, "d\omega^\vee"] \arrow[d, "\omega^\vee"] & \Omega_V|_{\mathcal{U}} \arrow[d, equal] \\ \mathbb{L}_Y^{\geq -1} & I/I^2 \arrow[r, "d"] & \Omega_V|_{\mathcal{U}} \end{tikzcd}$$

AND $d\omega|_{\mathcal{U}} = 0 \leftarrow \omega$ almost closed 1-form

Clearly $\omega = df$, $f : V \to \mathbb{C}$

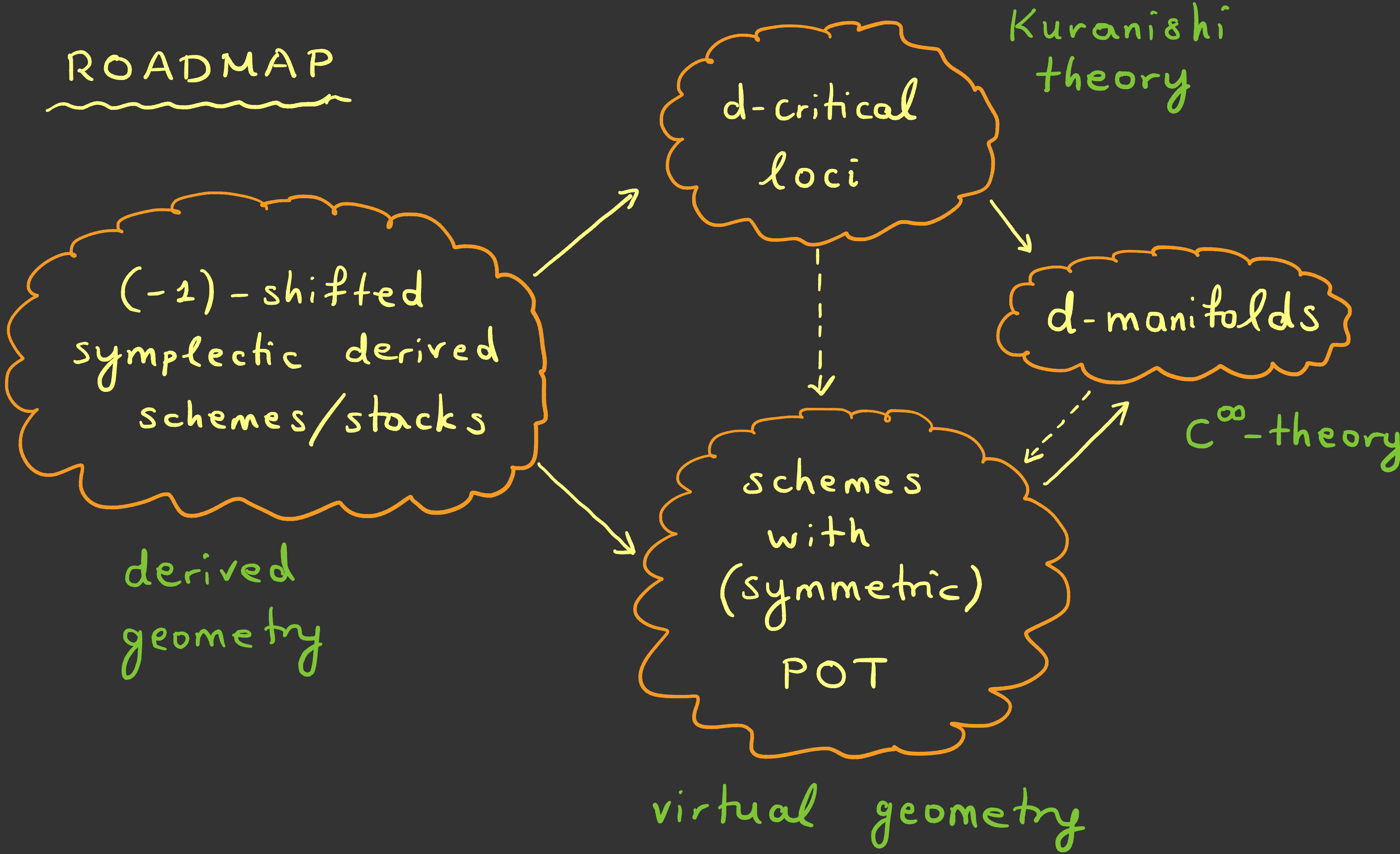
satisfies almost closedness.

Then $U = (df = 0) \subseteq V$ d-critical locus

<u>BUT</u>

- A priori, not all symmetric POTs look like this (locally). [Pand-Thomas]
- Not a global description, e.g. when Y proper.

ROADMAP



(-1)-shifted symplectic derived schemes/stacks
d-critical loci
d-manifolds
schemes with (symmetric) POT
Kuranishi theory
C∞-theory
derived geometry
virtual geometry

D-CRITICAL LOCI (Joyce, Kiem-Li)

Basic idea: Glue $U = (df = 0) \subseteq V$ together.

Definition (Joyce) Y scheme$/\mathbb{C}$.
$\exists$ canonical sheaf S_Y of $\mathbb{C}$-vector spaces s.t.
for $U \subseteq Y$ Zariski open & $U \hookrightarrow V$ closed
embedding with V smooth scheme, ideal I:

$$0 \longrightarrow S_Y|_U \longrightarrow \mathcal{O}_V/I^2 \xrightarrow{d} \Omega_V/I\cdot\Omega_V$$

$$f + I^2 \longmapsto df \bmod I\cdot\Omega_V$$

Example

$V = \mathbb{A}^1_z$

$\mathcal{U} = (z^n = 0) \subseteq V$ ← critical locus of $f = \dfrac{z^{n+1}}{n+1}$

$$S_{\mathcal{U}} = \left\{ a_0 + a_{n+1} z^{n+1} + \dots + a_{2n-1} z^{2n-1} + (z^{2n}) \right\} \cong \mathbb{C}^n$$

If $\quad f : V' \longrightarrow \mathbb{C}, \quad \mathcal{U} = (df = 0) \subseteq V',$

then $\quad f = a_{n+1} z^{n+1} + \dots, \quad a_{n+1} \neq 0$

$$f + (z^{2n}) = a_{n+1} z^{n+1} + \dots + a_{2n-1} z^{2n-1} + (z^{2n})$$

$$\pitchfork$$

$$H^0(S_{\mathcal{U}}) \implies$$ remember $a_{n+1}, \dots, a_{2n-1}$.

$$\frac{\partial^2 f}{\partial z^2} = (n+1)n\,a_{n+1} z^{n-1} + \cdots = (n+1)n\,a_{n+1} z^{n-1} \pmod{z^n}$$

$\Rightarrow$ symmetric POT remembers just a_{n+1}.

$$T_{v'}|_u \longrightarrow \Omega_{v'}|_u$$

$$\partial/\partial z \longmapsto (n+1)n\,a_{n+1}\,dz$$

Definition (Joyce) Y scheme$/\mathbb{C}$

d-critical structure: $s \in H^0(S_Y)$

(Y, s) is a **d-critical locus** (Zariski/étale/complex analytic)

if locally we have

- $U \subseteq Y$, $U = (df = 0) \subseteq V$
- $s|_U = f + I^2 \in \mathcal{O}_V/I^2$

$\Bigg\}$ d-critical chart (U, V, f)

Comparing d-critical charts

$y \in Y, \quad s \in H^0(S_Y).$

$$\left.\begin{array}{l} y \in U_1 = (df_1 = 0) \subseteq V_1 \\ y \in U_2 = (df_2 = 0) \subseteq V_2 \end{array}\right\} \text{d-critical charts}$$

(up to shrinking)

$$\begin{tikzcd} U_i \arrow[r, hook] & V_i \arrow[d, hook, "\phi_i"'] \arrow[dr, "f_i"] & \\ & W \arrow[r, "h"'] & \mathbb{C} \end{tikzcd}$$

locally closed

$f_1 = h|_{V_1}$

$f_2 = h|_{V_2}$

Slogan: Compare two charts by embedding into a common bigger chart.

<u>Intuition</u> If $\mathcal{U} = (df = 0) \subseteq V$

$$\begin{array}{c} \mathcal{U} = (df = 0) \subseteq V \\ \downarrow \\ \mathcal{U} = (dg = 0) \subseteq W \end{array}$$

& $\mathcal{U} = (dg = 0) \subseteq W$

then (analytically/étale locally)

Morse lemma $\longrightarrow$

$$V = \mathbb{C}^n$$

$$W = \mathbb{C}^{n+m}$$

$$g(z_1, \dots, z_{n+m}) = f(z_1, \dots, z_n) + z_{n+1}^2 + \dots + z_{n+m}^2$$

(related to stabilization, Thom-Sebastiani etc.)

D-critical loci are very flexible:

- D-critical Artin stacks,

  replace Zariski/étale by smooth.

- Equivariant d-critical loci,

  when reductive group $G \curvearrowright Y$

Canonical bundle & orientation of (Y, s)

$Y^{red} \subseteq Y$

For a d-critical chart (U, V, f)

have line bundle $K_V^{\otimes 2}\big|_{U^{red}} \cong \Lambda^{top}\left[T_V|_U \xrightarrow{\partial^2 f} \Omega_V|_U\right]$

Theorem (Joyce) These glue to canonical line bundle $K_{Y,s}$ on Y^{red}.

Definition An orientation on (Y,s) is a choice of square root $K_{Y,s}^{1/2}$.

Orientations are crucial in K-theory and other categorifications, e.g. motivic classes, perverse sheaves...

Example $\quad Y = (z^2 = 2yz = 0) \subseteq \mathbb{P}^2_{x,y,z}$

$$Y^{red} = (z=0) \cong \mathbb{P}^1_{x,y}$$

$$Y - (0:1:0) \cong \mathrm{Crit}\left(yz^2 : \mathbb{C}^2_{y,z} \to \mathbb{C}\right) \longrightarrow s = yz^2 + (z^2, 2yz)^2$$

$$Y - (1:0:0) \cong \{(x:y:0) : y \neq 0\} \cong \mathbb{C}_{\frac{x}{y}} \longrightarrow s = 0$$

computation shows that $K_{Y,s} \cong \mathcal{O}_{\mathbb{P}^1}(-5)$

thus Y, s is NOT orientable!

DERIVED ALGEBRAIC GEOMETRY (a la PTVV) (Lurie ..)

(some)
Motivation In deformation theory, for X a scheme$/\mathbb{C}$

& $x : \operatorname{Spec}\mathbb{C} \longrightarrow X$,

deformations of $x \in X$

$$\operatorname{Ext}^0_{\mathbb{C}}(\mathbb{L}_{X,x}, \mathbb{C}) \cong \operatorname{Hom}_*(\operatorname{Spec}\mathbb{C}[\varepsilon], X)$$

$$\operatorname{Ext}^1_{\mathbb{C}}(\mathbb{L}_{X,x}, \mathbb{C}) \overset{?}{\cong} \operatorname{Hom}_*(\Delta_1, X)$$

NO

no scheme Δ_1 works!

To resolve this, replace

$$\left(\text{commutative } \mathbb{C}\text{-algebras}\right)$$

by

$$\mathrm{cdga}_{\mathbb{C}}^{\leq 0} = \left(\begin{array}{c}\text{commutative differential} \\ \text{negatively graded } \mathbb{C}\text{-algebras}\end{array}\right)$$

objects are $A^{\bullet} = \cdots \xrightarrow{\delta} A^{-1} \xrightarrow{\delta} A^{0}$

satisfying $\delta^2 = 0$, $A^p \cdot A^q \subseteq A^{p+q}$

Koszul sign rule

$$\alpha_p \cdot \alpha_q = (-1)^{|p|\cdot|q|} \alpha_q \cdot \alpha_p$$

Leibniz rule

$$\delta(\alpha_p \cdot \alpha_q) = \delta\alpha_p \cdot \alpha_q + (-1)^{|p|} \alpha_p \, \delta\alpha_q$$

Example Koszul complex

$$\begin{array}{l} E \\ s \uparrow\downarrow \\ V \end{array}$$ vector bundle of rank r on smooth scheme

classical scheme $\swarrow$ $\mathcal{U} = (s=0) \subseteq V$

degree -1

$$0 \to \Lambda^r E^\vee \xrightarrow{\neg s} \Lambda^{r-1} E^\vee \to \cdots \to E^\vee \xrightarrow{s^\vee} \mathcal{O}_V \in \mathrm{cdga}_{\mathbb{C}}^{\leq 0}$$

Definition $$(\text{affine derived schemes}/\mathbb{C}) = (\mathrm{cdga}_{\mathbb{C}}^{\leq 0})^{op}$$

Thus an affine derived scheme is $\operatorname{Spec} A^\bullet$, for $A^\bullet \in \mathrm{cdga}_{\mathbb{C}}^{\leq 0}$.

surjection $A^\bullet \to H^0(A^\bullet) = A^0/\delta(A^{-1})$

induces $\operatorname{Spec} H^0(A^\bullet) \longrightarrow \operatorname{Spec} A^\bullet = X$

similar $\longleftrightarrow$ $Y^{red} \hookrightarrow Y$

classical truncation $\tau_0(X)$ of X

closed embedding

<u>Intuition</u> Derived scheme $X = \operatorname{Spec} A^\bullet$

$\updownarrow$

classical scheme $\tau_0(X)$

+ negative degree thickenings $H^i(A^\bullet)$, $i < 0$.

We can define a cotangent complex as follows:

Replace $A^\bullet$ by a quasi-free resolution, i.e.

$P^\bullet \longrightarrow A^\bullet$ quasi-isomorphism s.t.

graded algebra $P^\bullet$ is freely generated.

Then $\boxed{\mathbb{L}_{\operatorname{Spec} A^\bullet} \cong \Omega^1_{P^\bullet}}$ usual Kähler differentials

differential δ

Example Koszul complex $\cdots \Lambda^3 E^\vee \xrightarrow{\neg s} \Lambda^2 E^\vee \xrightarrow{\neg s} \underset{-1}{E^\vee} \xrightarrow{s^\vee} \underset{0}{\mathcal{O}_V}$

$A^\bullet = (\Lambda^\bullet E^\vee, \neg s)$ quasi-free, generated by $E^\vee$ in degree -1. $y_i \mapsto s_i$

$X = \operatorname{Spec} A^\bullet$

$$\mathbb{L}_{\operatorname{Spec} A^\bullet} \simeq \Omega^1_{A^\bullet}$$

If $E^\vee \cong \mathcal{O}_V y_1 \oplus \cdots \oplus \mathcal{O}_V y_r \longleftarrow$ deg -1 in $\Lambda^\bullet E^\vee$

$x_1, \ldots, x_n$ étale coordinates for V

then $\Omega^1_{A^\bullet} = \langle \underset{-1}{dy_j}, \underset{0}{dx_i} \rangle_{A^\bullet}$ with differential δ,

e.g. $\delta(dy_1) = \pm d(\delta y_1) = \pm ds_1 = \pm \sum_{i=1}^n \frac{\partial s_1}{\partial x_i} dx_i$

Observe that

$$\begin{tikzcd} & E \arrow[d] \\ X^{cl} = \tau_0(X) = (s=0) \subseteq & V \arrow[u, bend right, "s"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{L}_X|_{X^{cl}} \simeq \Omega^1_{A^\bullet}|_{H^0(A^\bullet)} \arrow[d] \simeq & {[\, T_V|_{X^{cl}}} \arrow[r, "ds^\vee"] \arrow[d, "s^\vee"] & {\Omega_V|_{X^{cl}} \,]} \arrow[d, equal] \\ \mathbb{L}^{\geq -1}_{X^{cl}} = & {[\, I/I^2} \arrow[r, "d"] & {\Omega_V|_{X^{cl}} \,]} \end{tikzcd}$$

is a perfect obstruction theory for X^{cl}!

Example of quasi-smooth derived scheme X.

As in the classical case, we can define open/closed/flat/smooth/étale morphisms between cdga's.

E.g. $A^\bullet \to B^\bullet$ étale

$$\Leftrightarrow \quad H^0(A^\bullet) \to H^0(B^\bullet) \text{ étale (usual definition)}$$

$$\& \quad H^i(A^\bullet) \otimes_{H^0(A^\bullet)} H^0(B^\bullet) \xrightarrow{\sim} H^i(B^\bullet)$$

QUESTION How to glue $\mathrm{Spec}\, A^\circ$ together?

ANSWER "Up to quasi-isomorphism & a lot of homotopical data"

Need model & ∞-categorical language,

VERY TECHNICAL.

$$
\begin{tikzcd}
\left(\text{commutative algebras}/\mathbb{C}\right) \arrow[rr, "\text{schemes}"] \arrow[drr, "\text{stacks}"] \arrow[ddrr, "\text{higher stacks}"'] \arrow[dd] & & \text{Sets} \arrow[d] \\
& & \text{Groupoids} \arrow[d] \\
\mathrm{cdga}^{\leq 0}_{\mathbb{C}} \arrow[rr, "\text{derived stacks}"'] & & \text{Simplicial Sets}
\end{tikzcd}
$$

satisfy (homotopical) descent

BACK TO MOTIVATION

Let $\mathbb{C}[\varepsilon_i] = \mathbb{C} \oplus \mathbb{C}[i]$ $\quad \varepsilon_i^2 = 0$

cdga in degrees 0 & $-i$.

$\Delta_i = \operatorname{Spec} \mathbb{C}[\varepsilon_i]$

i-th order infinitesimal disk

Then for X a derived scheme

$$\operatorname{Ext}^i_{\mathbb{C}}(\mathbb{L}_{X,x}, \mathbb{C}) \simeq R\operatorname{Hom}_*(\Delta_i, X)$$

for $x : \operatorname{Spec} \mathbb{C} \to X$.

[You can check this explicitly for $X = \operatorname{Spec} A^\bullet$]

SHIFTED SYMPLECTIC STRUCTURES

For simplicity, work in the affine case $X = \operatorname{Spec} A^\bullet$.

Global case: use smooth descent.

De Rham algebra

$$DR(A) = \underbrace{\bigoplus_{p=0}^{\infty}}_{\text{weight } p} \Lambda^p \Omega^1_A [p] = \bigoplus_{p=0}^{\infty} \underbrace{\bigoplus_{k=-\infty}^{0}}_{\text{degree } k-p} (\Lambda^p \Omega^1_A)^k [p]$$

two differentials

- d degree -1, weight 1
- δ degree 1, weight 0

Definition p-form of degree k

$$\omega^\circ \in (\Lambda^p \Omega^1_A)^k, \quad \delta\omega^\circ = 0 \in (\Lambda^p \Omega^1_A)^{k+1}$$

Two forms are equivalent if

$$\omega^\circ - \tilde{\omega}^\circ = \delta\alpha^\circ, \quad \alpha^\circ \in (\Lambda^p \Omega^1_A)^{k-1}$$

When $p=2$, a 2-form of degree k gives a skew-symmetric map

$$\omega^\circ : \mathbb{T}_A \longrightarrow \Omega^1_A[k]$$

If this is a quasi-isomorphism, then ω° non-degenerate.

Definition A closed p-form of degree k is

$$\omega = (\omega^0, \omega^1, \omega^2, \dots), \quad \omega^i \in (\Lambda^{p+i}\Omega^1_A)^{k-i}$$

s.t.

$$\delta\omega^0 = 0 \in (\Lambda^p\Omega^1_A)^{k+1}$$

$$d\omega^i + \delta\omega^{i+1} = 0 \in (\Lambda^{p+i+1}\Omega^1_A)^{k-i}$$

$\exists$ similar notion of equivalence.

Closed 2-form of degree k is k-shifted symplectic

if ω^0 is non-degenerate.

Remarks

- Being closed is not just a condition, it is extra data for a form.
- Closed 2-form = cocycle of negative cyclic homology of graded mixed complex DR(A)

Example Derived d-critical loci

Take $V = \mathbb{C}^n_x$ & A = Koszul complex of
$f : \mathbb{C}^n \to \mathbb{C}$ $E = \Omega_V$, $s = df$

$$X = \operatorname{Spec} A = R\mathrm{Crit}(f)$$

$$\mathbb{L}_X = \Omega^1_A = \langle \underset{\text{deg } -1}{dy_i}, \underset{\text{deg } 0}{dx_i} \rangle$$ where $y_i \longleftrightarrow dx_i$ as frame of Ω_V.

with $$\delta(dy_i) = \pm d(\delta y_i) = \pm d\left(\frac{\partial f}{\partial x_i}\right) = \pm \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j$$

Define $\omega^0 = \sum_i dy_i \wedge dx_i \in (\Lambda^2 \Omega^1_A)^{-1}$ 2-form deg = -1

$\omega^1 = \omega^2 = \dots = 0$

Then $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ is (-1)-shifted symplectic!

Why?

- $\delta\omega^0 = 0 \leftarrow \frac{\partial^2 f}{\partial x_i \partial x_j}$ symmetric, but $dx_i \wedge dx_j$ skew-symmetric
- $d\omega^0 + \delta\omega^1 = 0$
  $d\omega^1 + \delta\omega^2 = 0$ $\leftarrow$ obvious
  $\vdots$
- $\omega^0 : \mathbb{T}_A \to \Omega^1_A[1]$, $\frac{\partial}{\partial x_i} \mapsto dy_i$, $\frac{\partial}{\partial y_i} \mapsto dx_i$ clearly non-degenerate

MAJOR

Theorem (PTVV) X projective Calabi-Yau of dimension d.

Then the (derived) stack $\mathcal{M}$ of perfect complexes on X admits a $(2-d)$-shifted symplectic structure.

$d=2$ X K3, $\mathcal{M}$ 0-shifted symplectic $\rightarrow$ hyperkähler moduli

$d=3$ X CY3, $\mathcal{M}$ (-1)-shifted symplectic $\rightarrow$ DT3 theory

$d=4$ X CY4, $\mathcal{M}$ (-2)-shifted symplectic $\rightarrow$ DT4 theory

MAJOR

Theorem (Joyce ét al) Every k-shifted symplectic derived scheme (Artin stack) has a canonical Zariski (smooth) local model, called Darboux form.

(mirroring the usual Darboux theorem in symplectic geom.)

$\boxed{k=-1}$ Darboux model = derived d-critical locus $\mathrm{RCrit}(f)$

Corollary Moduli spaces of sheaves on CY3-folds are d-critical loci, as defined earlier!

Example Hilbert scheme of points on $\mathbb{A}^3$

$$(\mathbb{A}^3)^{[n]} = \left\{ v \in \mathbb{C}^n,\ A, B, C \in \mathrm{Mat}_{n\times n}(\mathbb{C}) : \begin{array}{l} [A,B] = [B,C] = [C,A] = 0 \\ \mathbb{C}[A,B,C]\cdot v = \mathbb{C}^n \end{array} \right\}$$

$=$ critical locus of

$$f : \mathbb{C} \times \mathrm{Mat}^3_{n\times n} \longrightarrow \mathbb{C}$$

$$(v, A, B, C) \longmapsto \operatorname{tr} A[B,C].$$

$\boxed{k=-2}$ A (-2)-shifted symplectic derived scheme X is locally determined by:

$$\begin{tikzcd} E \arrow[d] \\ V \arrow[u, bend left, "s"] \end{tikzcd}$$

E vector bundle, V smooth $+$ $q: E \otimes E \to \mathcal{O}_V$ non-degenerate quadratic form

s.t. s isotropic, i.e. $q(s,s)=0$.

Locally $X^{cl} = \tau_0(X)$ is $\mathcal{U} = (s=0) \subseteq V$.

Remark All the usual notions of symplectic geometry, e.g. Lagrangians, Hamiltonians, carry over to this setting.

C^∞-ALGEBRAIC GEOMETRY & D-MANIFOLDS

(Joyce)

Definition A C^∞-ring is a set $\mathcal{C}$ with operations

$\Phi_f : \mathcal{C}^n \to \mathcal{C}$ for all $n \geq 0$, $f : \mathbb{R}^n \to \mathbb{R}$ smooth satisfying:

- For $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$ and $h = g(f_1, \dots, f_m)$
  $g : \mathbb{R}^m \to \mathbb{R}$

  we have $\Phi_h(c_1, \dots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \dots, c_n), \dots, \Phi_{f_m}(c_1, \dots, c_n))$

- $\pi_j : \mathbb{R}^n \to \mathbb{R}$, $\Phi_{\pi_j}(c_1, \dots, c_n) = c_j$
  $(x_1, \dots, x_n) \mapsto x_j$

A C^∞-ring is an $\mathbb{R}$-algebra:

$$c_1 + c_2 = \Phi_f(c_1, c_2) \quad , \quad f(x,y) = x+y$$

$$c_1 \cdot c_2 = \Phi_g(c_1, c_2) \quad , \quad g(x,y) = xy$$

$$\lambda \cdot c = \Phi_{\lambda'}(c) \quad , \quad \lambda'(x) = \lambda x$$

<u>Examples</u> X smooth manifold.

Then $C^0(X) = \{ \text{continuous } X \to \mathbb{R} \}$ are C^∞-rings.

$$C^\infty(X) = \{ \text{smooth } X \to \mathbb{R} \}$$

Points of Spec $\mathcal{C}$ are $\mathbb{R}$-algebra morphisms $x: \mathcal{C} \to \mathbb{R}$.

(Joyce)

Definition C^∞-scheme is a ringed space

locally modelled on Spec $\mathcal{C}$.

(Usually) take $\mathcal{C}$ finitely generated,

i.e. $\mathcal{C} = C^\infty(\mathbb{R}^n)/I$.

Have usual properties (proper etc.)

& theory of sheaves, replicating algebraic counterparts.

Definition (Joyce)

d-space $(X, \mathcal{O}_{X'}, \mathcal{E}_X, i_X, j_X) = \underline{X}$

- $(X, \mathcal{O}_X)$ C^∞-scheme
- $(X, \mathcal{O}_{X'})$ C^∞-scheme
- $\mathcal{E}_X$ quasi-coherent sheaf on $(X, \mathcal{O}_X)$
- $\mathcal{E}_X \xrightarrow{j_X} \mathcal{O}_{X'} \xrightarrow{i_X} \mathcal{O}_X \to 0$ exact

  with $I_X = \ker i_X$ square-zero.

Have

$$X \hookrightarrow X'$$

$$\phi_X : \mathcal{E}_X \longrightarrow I_X \xrightarrow{d} \Omega_{X'}|_X = \mathcal{F}_X \quad , \qquad \mathcal{F}_X \xrightarrow{\psi_X} \Omega_X$$

and (virtual) cotangent sheaf

$$\left[\mathcal{E}_X \xrightarrow{\phi_X} \mathcal{F}_X \right]$$

(1-)Morphisms : $\underline{X} \to \underline{Y}$ are as expected.

2-Morphisms : $\eta : f \Rightarrow g$ are $\eta : \mathcal{F}_Y|_X \longrightarrow \mathcal{E}_X$

playing the role of homotopy.

Definition (Joyce) Let V manifold of dim $n+r$

$E \to V$ C^∞-vector bundle of rank r

$s \in C^\infty(E)$ C^∞-section

Then $S_{V,E,s}$ is called a principal d-manifold

where the data is

$$C^\infty(E^\vee) / (s)\cdot C^\infty(E^\vee) \longrightarrow C^\infty(V)/(s)^2 \longrightarrow C^\infty(V)/(s) \longrightarrow 0$$

The underlying topological space is $(s=0) \subseteq V$.

d-manifold = d-space locally equivalent to $S_{V,E,s}$.

Cotangent sheaf of $S_{V,E,s}$

$$\left[C^\infty(E^\vee) \big/ (s)\cdot C^\infty(E^\vee) \xrightarrow{ds^\vee} T^*V \big/ (s)\cdot T^*V \right]$$

Same form as POT !

All usual notions of differential geometry translate to this setting, e.g. immersions, submersions, partitions of unity ...

In particular:

Theorem (Joyce) X compact d-manifold.

Then
- $\exists$ closed embedding $X \hookrightarrow \mathbb{R}^N$. (Whitney embedding theorem)
- $X \simeq S_{E,V,s}$ for some E, V, s.

Finally, an orientation of X is a trivialization of the (virtual) canonical bundle.

compact, oriented d-manifolds have virtual fundamental classes.

Theorem (Joyce) $\exists$ functor

$$\begin{pmatrix} \text{schemes}/\mathbb{C} \\ \text{with POT} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{oriented d-manifolds} \end{pmatrix}$$

Application to DT4 invariants (Joyce - Borisov)

M = moduli scheme parametrizing stable sheaves on CY 4-fold

Then M is the classical truncation of a (-2)-shifted symplectic derived scheme.

Thus locally M looks like

$$\begin{tikzcd} & E \arrow[d] \\ U = (s=0) \subseteq & V \arrow[u, bend right, "s"'] \end{tikzcd} \quad , \quad \begin{array}{l} E \otimes E \xrightarrow{q} \mathcal{O}_V \\ q(s,s) = 0. \end{array}$$

Idea Split $E = E^+ \oplus E^-$, $s = (s^+, s^-)$

maximal positive & negative definite $\mathbb{R}$-sub-bundles

Then S_{V, E^-, s^-} glue together to give a d-manifold M^{dm}.

Orientation of M $\rightsquigarrow$ orientation of M^{dm}.

Thus get oriented d-manifold M^{dm}.

If M proper, M^{dm} is compact.

Definition The DT4 invariant of M is defined

as $[M^{dm}]^{vir} \in H_*(M)$

and is well-defined & deformation-invariant.